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A Theory of Money and Financial Institutions. Part 25. A Closed Economy with Exogenous Uncertainty, Different Levels of Information, Money, Futures and Spot Markets

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PART 25

A CLOSED ECONOMY WITH EXOGENOUS UNCERTAINTY, DIFFERENT LEVELS OF INFORMATION, MONEY, FUTURES AND SPOT MARKETS

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PART 25

A CLOSED ECONOMY WITH EXOGENOUS UNCERTAINTY, DIFFERENT LEVELS OF INFORMATION, MONEY, FUTURES AND SPOT MARKETS*

by

Pradeep Dubey and Martin Shubik

1. MODELS OF TRADE WITH MONEY

1.1. Money Games

In several previous publications [1, 2, 3] a model was constructed of a closed trading economy using a money. The characterizing feature of the money is that it is used for all trades. The trading mechanism considered was based on a Cournot type of market with price being determined by "the quantity of money chasing a quantity of goods."

There are other models of trade in money which could be constructed where sellers name a price for their goods. However all models with money have as their central feature that the use of a commodity (real or fiat) as a money may impose a restriction on the strategy spaces of the traders. The explicit market structures and rules of trade with money may vary but each one may be regarded as imposing a restriction in some form on the set of trades feasible in the noninstitutional "barter" trading model of exchange. [4]

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The distinction between a money and a nonmoney is strategic. The distinction between a commodity money and a fiat money can be made in terms of its presence in or absence from the utility functions of the traders were the money to be demonetized. A commodity money such as "salt bars" would maintain a value even if it were demonetized. A fiat money might have some worth as a collector's item if it were demonetized, but that is all.

A Money Game is a model of trade (and production) in which there is a distinguished commodity (real or fiat) which is used as a money; and in which complete rules for the formation of price, as a result of individual strategies, are given.

The explicit definition of the games does not necessarily imply the type of solution concept to be used. In the papers noted above and in this paper we limit our enquiry to the noncooperative equilibrium and its relationship with the competitive equilibrium.

1.2. Compound Futures Markets

Arrow and Debreu [5, 6] in their treatment of uncertainty in a context of general equilibrium introduced the concept of trade in contingent commodities. Their model was essentially static. Although one could argue that that both time and uncertainty can be accounted for by enlarging the number of commodities by differentiating them both with respect to time and contingencies; the analysis presented treats exchange and production as though all decisions are made in a single one period game in strategic form.

One interpretation of trade in contingent commodities is that the traders are buying lotteries rather than commodities, i.e. they are purchasing claims to goods under certain circumstances rather than goods. When we attempt to model trade with uncertainty as a game there are several different ways in which a Cournot type of model can be constructed. They serve to distinguish differences between spot and futures market trading.

The first market we consider is one with compound futures trading only. This is the market that has been investigated by Dubey and Shubik in a previous paper [7]. The other two markets are respectively one with pure lotteries trading only and one which can be interpreted as having both spot and futures markets. The general mathematical structure of the first market has been given in the paper by Dubey and Shubik. The general structure of the other two is given in Section 2 where we discuss existence of noncooperative equilibria. In the remaining parts of this section we use a specific example to illustrate the differences in the three models. The simplest example requires two traders, with one consumer good, two "states of the economy" and a money. For the purposes of illustration we use a commodity money which enters the utility function as a linear additive term. In the general treatment in Section 3 this plays no role.

Before we calculate an example and examine the general markets a simple illustration shows the three different markets. Suppose that there are two traders, there is one commodity "oranges" and two states of nature "rain" or "shine." There are 10 oranges for sale "if rain" and 20 "if shine."

We may construct three types of markets to sell the oranges or futures in oranges.

(1) Compound Futures

10 with probability	ρ
20 with probability	$(1-\rho)$

There is one black box whose contents will be 10 oranges with probability ρ and 20 with probability $(1-\rho)$. Traders bid on the box without necessarily knowing its contents. We call this a compound future as it pays out different amounts for various states.

(2) Pure Futures

10 with probability	ρ
0 with probability	$(1-\rho)$

20 with probability	$(1-\rho)$
0 with probability	ρ

Here there are as many black boxes as contingent commodities. Each box has a payout for one state or it is empty. A trader who bids on a box which turns out to be empty will have bought a future which is *ex post* worthless. If he has the information prior to trade that certain boxes are empty he can avoid bidding on them. In the Arrow and Debreu treatments the individuals are assumed to have the same lack of knowledge about the contents of the boxes.

(3) Spot and Futures Markets

10 oranges with certainty

10 oranges with probability	$(1-\rho)$
0 with probability	ρ

For each commodity there may be a positive amount that will be present regardless of state. Thus in this example 10 oranges can be offered in a "transparent box," i.e. everyone knows that there are 10 oranges for sale independent of state. There are more oranges which are state dependent and those can be sold in simple or compound futures markets. Here they are sold in a simple futures market.

A Terminological Difficulty

A "futures" contract in ordinary economic life is a contract for the delivery of some commodity at a future date. As Debreu [8] has noted an "agent who buys a bushel of No. 2 Red Winter Wheat available in Chicago at date t _in any event_ buys in fact as many commodities as there are events at t ." This statement is not quite complete. In actuality implicit in dealings in futures is a _force majeure_ clause which recognizes the possibility that some disaster not under the control of the seller of the future will make it impossible to meet his commitments, thus even an ordinary futures contract does not guarantee delivery.

A spot market refers to a market for a commodity that is known to exist and is available for immediate delivery. In a one period model such as the one described above it is not easy to make the natural distinction between a spot and a futures market. We can however imagine an individual taking actions at the start of a period which result in deliveries at the end of the period. If he is absolutely certain that he will get the delivery we might call that market a "spot market" or a "guaranteed futures market." Those deliveries which are state dependent can be regarded as arising from simple or compound futures markets.

An Example

We illustrate the different markets for an instance with nonsymmetric information. Trader 1 cannot distinguish the two states of the economy whereas Trader 2 can. Let each trader have a utility function of the form:

$$U_i = \log q^i + m^i$$

where q^i is the amount of the consumer good obtained by i and m^i is the residual amount of the commodity money at the end of trade.

Suppose that in state 1 trader 1 has A_1 units of the commodity and M of money. In state 2 trader 1 has A_2 units of the commodity and M of money while trader 2 has B_1 and B_2 units and M of money in the two states respectively. The traders are required to deposit all of their commodity in the market for sale. They then bid in money to buy it and receive an income from their ownership claims. The information state is shown in Figure 1. Nature moves first to determine which state exists. The first trader must make his bid $b' \leq M$, without

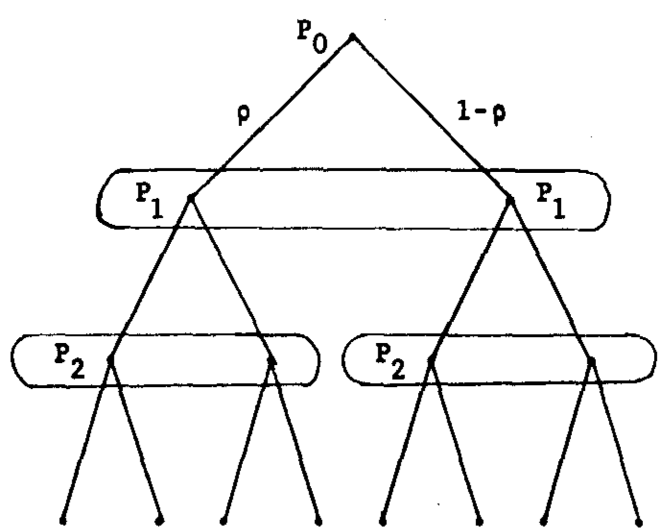

FIGURE 1

knowing the state of the economy. The second trader has a strategy with two components; as he knows the state of the system he bids b_1^2 under state 1 and b_2^2 under state 2.

Let $\alpha_i = A_i/(A_i + B_i)$.

There is one compound futures market for the consumer commodity and in terms of the str tegies the payoffs of the traders are:

$$(1) \qquad \Pi_1 = \rho \log \frac{b^1}{b^1 + b_1^2} + (1-\rho)\log \frac{b^1}{b^1 + b_2^2} - b^1 + M + \rho\alpha_1(b^1 + b_2^2)$$

$$+ (1-\rho)\alpha_2(b^1 + b_2^2) + K$$

$$(2) \qquad \Pi_2 = \rho \log \frac{b_2^2}{b^1 + b_2^2} + (1-\rho)\log \frac{b_2^2}{b^1 + b_2^2} - \rho b_1^2 - (1-\rho)b_2^2 + M$$

$$+ (1-\alpha_1)(b^1 + b_2^2) + (1-\rho)(1-\alpha_2)(b^1 + b_2^2) + K$$

where $K = \rho \log (A_1 + B_1) + (1-\rho)\log(A_2 + B_2)$. We can see from (1) and (2) that the nonsymmetry in information is translated into a restriction on the strategy spaces for the first trader. This is discussed in Section 3.

1.3. Simple Futures Markets

We may consider the following "game." All individuals are given ownership claims to contingent commodities. Each individual is required to offer these claims for sale in a futures market. Each individual obtains the actual commodities from the ownership of valid contracts at the end of the period. There are no "spot markets"; the traders buy only paper claims not goods.

The implication of complete information for a trader is that he knows in advance of moving which futures contracts will pay off and which are worthless. In this model all traders have a larger set of moves than do the traders in the compound futures market model. In the compound futures markets there were m trading posts, here there are mk posts for simple futures markets. The strategy sets of the traders with more information are considerably larger than for those with less as they select their moves contingent upon their extra knowledge.

Using the same economic background as in the previous example we note that the strategy for trader 1 is a pair of bids (b_1^1, b_2^1) whereas the strategy for trader 2 is two pairs of bids. The first contingent on event 1 and the second upon event 2. Let b_{ij}^2 stand for trader 2's bid in market i if contingency j occurs. The payoffs can be written

$$
\begin{aligned}
(3)\qquad \Pi_1 = {} & \rho \log\left(\frac{b_1^1}{b_1^1 + b_{11}^2}\right) + (1-\rho)\log\left(\frac{b_2^1}{b_2^1 + b_2^2}\right) - b_1^1 - b_2^1 + M \\
& + \rho\alpha_1(b_1^1 + b_{11}^2) + (1-\rho)\alpha_1(b_1^1 + b_{12}^2) + \rho\alpha_2(b_2^1 + b_{21}^2) \\
& + (1-\rho)\alpha_2(b_2^1 + b_{22}^2) + K
\end{aligned}
$$

$$
\begin{aligned}
(4)\qquad \Pi_2 = {} & \rho\left\{\log\left(\frac{b_{11}^2}{b_1^1 + b_{11}^2}\right) - b_{11}^2 + M + (1-\alpha_1)(b_1^1 + b_{11}^2)\right\} \\
& + \rho\left\{0^* - b_{12}^2 + M + (1-\alpha_1)(b_2^1 + b_{12}^2)\right\} \\
& + (1-\rho)\left\{0^* + b_{21}^2 + M + (1-\alpha_2)(b_1^1 + b_{21}^2)\right\} \\
& + (1-\rho)\left\{\log\left(\frac{b_{22}^2}{b_2^1 + b_{22}^2}\right) - b_{22}^2 + M + (1-\alpha_2)(b_2^1 + b_{22}^2)\right\} + K .
\end{aligned}
$$

1.4. Spot and Futures Markets

The uncertainty of the traders has two components; their uncertainty about total supplies and their uncertainty about their individual ownership claims. There will be a minimum amount of each commodity which will exist with certainty. This amount can be sold in a spot market. The amount which will appear uncertain to at least some individual will not be sold directly but in a futures market. Thus for m commodities and k states there will in general be m spot markets and $(k-1)m$ simple futures markets at most.

1.5. A Comparison of Models

Suppose for the sake of simplicity that $A_1 = B_2$ and $A_2 = B_1$ then we may easily solve the models in 1.2 and 1.3 for the limiting (equal treatment) equilibrium of the markets under replication. We obtain from the first order conditions from (1) and (2) and replication

Compound Futures Markets

(5) $$b^1 = 1$$

$$b_1^2 = 1\ ,\quad b_2^2 = 1$$

and similarly from (3) and (4).

---

*We define here as 0 the expected utility that trader 2 obtains from investing in a futures market that he knows will have no deliveries. It follows immediately that $b_{12}^2 = 0$ and $b_{21}^2 = 0$ will be his optimal strategies.

<u>Simple Future Markets</u>

$$(6) \qquad b_1^1 = 1 \qquad\qquad b_2^1 = (1-\rho)$$

$$b_{11}^2 = 1 \qquad\qquad b_{22}^2 = 1$$

$$b_{21}^2 = 0 \qquad\qquad b_{12}^2 = 0 \ .$$

Hence the payoffs with spot markets are:

$$\Pi_1 = \log \frac{1}{2} + M + \log(A+B)$$

$$\Pi_2 = \log \frac{1}{2} + M + \log(A+B) \ .$$

The payoffs with futures markets are

$$\Pi_1 = \rho \log \left(\frac{\rho}{1+\rho}\right) + (1-\rho)\log\left(\frac{1-\rho}{2-\rho}\right) + M + \log(A+B)$$

$$\Pi_2 = \rho \log \left(\frac{1}{1+\rho}\right) + (1-\rho)\log\left(\frac{1}{2-\rho}\right) + M + \log(A+B) \ .$$

The advantage is clearly to the trader with information when there are simple futures markets. When $\rho = 0$ or 1 there is certainty and the two cases become the same.

It is a simple exercise to calculate the outcome for this game with both a spot and futures market.

2. PREFERENCE, UTILITY AND PROBABILITY

Suppose that there are $m+1$ commodities in a society and that the $m+1^{st}$ is used as a money. We wish to stress that from the point of view of the general development of the theory of noncooperative money games it makes no essential difference if the utility functions are of the form

$$\text{(a)} \quad U_i = \varphi_i(x_1^i, \ldots, x_m^i, x_{m+1}^i)$$

$$\text{(b)} \quad U_i = \varphi_i(x_1^i, \ldots, x_m^i) + \lambda_i x_{m+1}^i$$

or

$$\text{(c)} \quad U_i = \varphi_i(x_1^i, \ldots, x_m^i) \ .$$

The form (a) is that for a general commodity money; (b) for a "linearly transferable utility commodity" and (c) for a fiat money. The form (b) is the easiest to use to provide easy-to-calculate examples. The form (c) appears to be most natural in multistage models. Our existence proofs hold for any of these forms. This point has been discussed elsewhere [9]. It must be stressed that in many partial equilibrium models money has been placed in the utility function in a <u>post hoc ergo propter hoc</u> manner. This is not done here. If it appears at all in the utility functions this merely reflects its worth as a commodity <u>not</u> its strategic value as a money. Its strategic value as a money must be deduced as part of the solution, not invoked as an assumption.

2.2. Utility Functions and Uncertainty

Suppose that there are m commodities and k states of nature. Following Arrow and Debreu we can consider an economy with km contingent commodities. Given these commodities we may consider a utility function for individual i of the form

$$\text{(d)} \qquad u^i = \varphi_i(x_{11}^i, x_{21}^i, \ldots, x_{mk}^i)$$

with no further restrictions imposed than those for classical indifference curves.

Suppose that the different states of nature may occur with objective probabilities $(\rho_1, \rho_2, \ldots, \rho_k)$ where $\sum_{j=1}^{k} \rho_j = 1$. We might argue that the utility function should be restricted to the form:

$$\text{(e)} \qquad u_i = \sum_{j=1}^{k} \rho_j \varphi_i(x_{1j}^i, \ldots, x_{mj}^i)$$

or

$$\text{(f)} \qquad u_i = \sum_{j=1}^{k} \rho_j \varphi_{ij}(x_{1j}^i, \ldots, x_{mj}^i) \,.$$

In (e) it is implicitly assumed that the utility for "oranges when the sun shines" is the same as for "oranges when it rains." In (f) the possibility that the utility for the same commodity in different states might differ is taken into account.

An alternative interpretation of uncertainty and the utility function can be made in terms of subjective probabilities. Each individual might have personal probability estimates of the occurrence of the different states. These probabilities are reflected in a utility function of the form:

$$\text{(g)} \qquad u_i = \sum_{j=1}^{k} \rho_j^i \varphi_{ij}(x_{1j}^i, \ldots, x_{mj}^i) ,$$

where ρ_j^i is i's subjective probability that event j will occur.

2.3. The Radner Model

In his paper on equilibrium under uncertainty Radner [9] has considered an extension of the general equilibrium model to cases where individuals have different information about uncertain events. His model is not formulated as a game but may be related to a noncooperative game.

A two person, one commodity, three state example serves to illustrate the Radner approach. Consider an economy with the information structure as illustrated in Figure 2. Trader 1 cannot distinguish state 1 from 2; and Trader 2 cannot distinguish state 2 from 3.

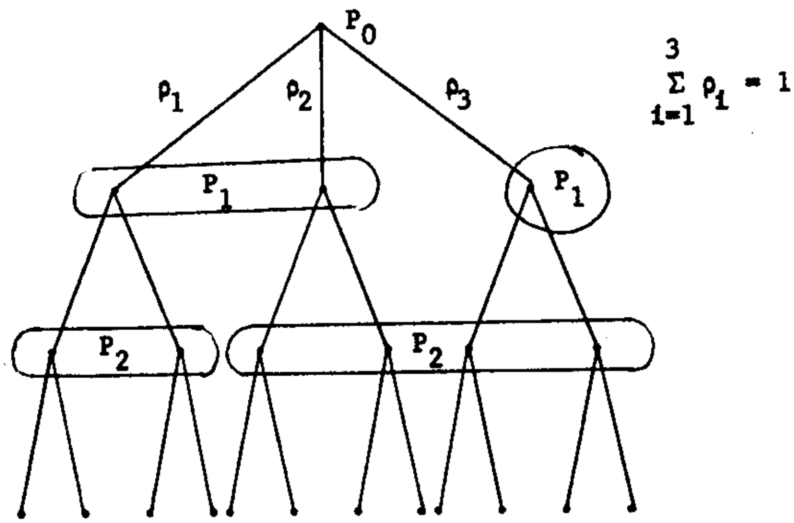

FIGURE 2

The referee knows that the endowments of the traders in the different states are respectively (w_1^1, w_2^1, w_3^1) and (w_1^2, w_2^2, w_3^2) . Suppose that there are three market prices p_1 , p_2 , and p_3 .

The wealth of trader i is $p \cdot w^i = \sum_{j=1}^{3} p_j w_j^i$. Suppose trader 1 has a utility function of the form $\varphi_1(x_{12}^1, x_3^1)$ and trader 2, $\varphi_2(x_1^2, x_{23}^2)$ where x_{12}^1 and x_{23}^2 are composite commodities. Define two composite prices $p_{12} = p_1 + p_2$ and $p_{23} = p_2 + p_3$. Each individual may maximize his welfare subject to his budget constraint so that

$$p_{12}x_{12}^1 + p_3x_3^1 = p \cdot x^1 \leq p \cdot w^1 \text{ and similarly for 2 .}$$

The probabilities ρ_1 , ρ_2 , ρ_3 are implicitly accounted for in the utility functions. We might have this explicit in utility functions of the form:

$$\rho_{12}\varphi_1(x_{12}^1) + \rho_3\psi_1(x_3^1) \quad \text{and} \quad \rho_1\psi_2(x_1^2) + \rho_{23}\varphi_2(x_{23}^2) \; .$$

More specifically suppose the three states occur with probabilities (1/3, 1/3, 1/3) and that the utility function for Trader 1 is:

$$u_1 = \frac{2}{3} \log x_{12}^1 + \frac{1}{3} \log x_3^1$$

and for Trader 2 is:

$$u_2 = \frac{1}{3} \log x_1^2 + \frac{2}{3} \log x_{23}^2 \; .$$

Let the endowments by (2,2,6) and (6,2,2) respectively. It is easy to show that a price system of $p_1 = p_3 = 1/3$ and $p_{12} = p_{23} = 1$ clears these markets. This is discussed further in 2.4.

It is important to note that the Radner model requires that individuals who cannot distinguish between states must have the same endowment in each state among which they cannot distinguish. This implies that even after trade the traders never find out about states that they could distinguish. The Radner analysis does not cover the case where endowments differ in states which cannot be distinguished before decision, but may be distinguished after. Changing the endowments in the example above to (2,16,6) and (6,16,2) there is no solution suggested by Radner.

2.4. The Radner Model as a Noncooperative Game

The models in Section 1 do not include the Radner model. Can we describe a noncooperative game which has a limiting equilibrium under replication the Radner result? It is possible to construct such a game, but it has the unfortunate feature that the number of futures markets may become enormous. Suppose that there are m commodities and k states, then the number of markets can vary from as low as m to as high as $2^{mk} - 1$. Where as a result a nonsymmetric information there is a need for futures markets over aggregates of every combination of states.

Reconsidering the example in 2.3 first for two traders and then for many traders of two types we may consider the following game:

All traders assign all of their potential endowments to a market mechanism. Many combinations of futures contracts are offered including possibly unconditional delivery which we may regard as the equivalent of a spot market.

Traders bid for contracts which must be no more refined than their perceptions; i.e. if a trader buying oranges cannot distinguish between "oranges if it rains" and "oranges if the sun shines" and if he will not be able to make this distinction after the markets have cleared then he does not distinguish ever between a market which sells him futures in certain individual or compound states.

A comparison of the two sets of initial conditions given in the example in 2.3 shows the contrasting game. Although the decision structure is decentralized the computational requirements on the market mechanism or clearing houses may be large.

There are 3 commodities plus a commodity money which enters the utility functions as a linear separable term. The other commodities are:

1. oranges if it rains
2. oranges if it snows
3. oranges if the sun shines

Trader 1 does not distinguish states 1 or 2

Trader 2 does not distinguish states 2 or 3.

This lack of distinction holds before they have to act. After they act we must distinguish two cases (a) where they cannot distinguish cases *ex post*, (b) where there is an individual *ex post* distinction.

Consider that the market will offer 4 types of contract. They are

1. oranges if it rains
2. oranges if the sun shines
3. oranges if it rains or snows
4. oranges if the sun shines or it snows

<u>The Black Boxes</u>

The four contracts offered in the Radner markets are shown below.

6 oranges in state 1 with probability 1/3 0 with probability 2/3	2 oranges in state 2 or 3 with probability 2/3 0 with probability 1/3	2 oranges in state 1 or 2 with probability 2/3 0 with probability 1/3	6 oranges in state 3 with probability 1/3 0 with probability 2/3

They are specially compounded "simple futures" in the sense that although the market for oranges in state 2 or 3 pays off for state 2 or state 3 it does not for state 1. By Radner's hypothesis the number of oranges available in each set of unrecognizable states is the same, then states 2 and 3 are indistinguishable <u>ex post</u> and <u>ex ante</u>.

We can formally write down two payoff functions restricting the first trader to markets 12 and 3 and the second to markets 23 and 1 and obtain Radner's results from a noncooperative game; however, the modelling appears to be somewhat forced and does not fit that of a satisfactory strategic game. We have to assume that the individuals buy in different futures markets when they have different information and a central marketing agency works out how to stock these different markets.

Any of our three markets in Section 1 are well defined for handling nonsymmetric information in general and they each appear to be far less complex than the Radner model.

3. THE EXISTENCE OF A NONCOOPERATIVE EQUILIBRIUM WITH NONSYMMETRIC INFORMATION

3.1. The Existence Proof

For a positive integer r we shall denote by I_r the set $\{1, \ldots, r\}$, by E^r the Euclidean space of dimension r, and by Ω^r the non-negative orthant of E^r.

Let

I_n = the set of traders

I_{m+1} = the set of commodities

I_s = the set of "states of nature," which occur with positive probabilities $\rho_1, \ldots, \rho_s$.

The initial allocation of trader i is a vector $a^i \in \Omega^{s(m+1)}$, where ${}^k a_j^i$ is the amount of commodity j available to i in state k (for $j \in I_{m+1}$, $k \in I_s$). Note ${}^k a_{m+1}^i$ represents the money held by i in state k. It will be convenient to consider $\Omega^{s(m+1)}$ with an axis for each commodity in each state. By axis jk we will mean the axis for the j^{th} commodity in the k^{th} state. Let us assume that any good j will have different utilities in different states; and let ${}^k u^i : \Omega^{m+1} \to \Omega^1$ be a continuous, non-decreasing, concave function which gives the utility of commodity bundles in state k. We can think of ${}^k u^i$ as a function from $\Omega^{s(m+1)}$ to Ω^1 where ${}^k u^i(x)$ for $x \in \Omega^{s(m+1)}$ depends only on* ${}^k x_j$, $j \in I_{m+1}$. Then the utility function of i, $u^i : \Omega^{s(m+1)} \to \Omega^1$ is

$$u^i(x) = \sum_{k \in I_s} \rho^k \, {}^k u^i(x) .$$

---

* ${}^k x_j$ is the projection of x on the axis jk.

We shall say that trader i "desires" good j in state k if ${}^{k}u^{i}(x)$ is an increasing function of ${}^{k}x_j$, for any fixed choice of the other variables.

Finally, to complete the basic data of the market, we must specify the "information" of every trader i. This will consist of a partition $\mathcal{P}^i$ of I_s into non-empty sets, where the states in any member of i are precisely those that he cannot distinguish. For any $k \in I_s$ let $\mathcal{P}^i(k)$ be the set of all those states in I_s that i cannot distinguish from k. We will write* (for ℓ and k in I_s) $\ell \sim_i k$ if $\ell \in \mathcal{P}^i(k)$; $\not\sim_i$ will mean "not $\sim_i$." If $\min\limits_{\ell \in \mathcal{P}^i(k)} {}^{\ell}a^i_{m+1} > 0$ we will say that trader i is "k-moneyed."

When we omit a subscript or superscript, and use a bar, it will denote summation over the indexing set of the omitted script. Thus for $x^i \in \Omega^{sm}$, $\overline{{}^{k}x}^{i} = \sum\limits_{j \in I_m} {}^{k}x^i_j$ and $\overline{x}^i = \sum\limits_{k \in I_s} \sum\limits_{j \in I_m} {}^{k}x^i_j$, etc.

To cast the rules of the market in the form of a game, we need to define the strategy sets S^i for $i \in I_n$.

Let ${}^{k}_{\ell}b^i_j$ denote i's bid, in state k, at the market for the contingent commodity $j\ell$ ($j \in I_m$, $\ell \in I_s$). Then

$$
\begin{aligned}
S^i = \{b^i \in \Omega^{sm^2} : \; & {}^{k}_{\ell}b^i = 0 \text{ if } k \not\sim_i \ell, \\
& {}^{k}_{\ell}b^i = {}^{k'}_{\ell}b^i \text{ if } k \sim_i k' \text{ and } k \sim_i k' \\
& \overline{{}^{k}b^i} \le \min_{\ell \in \mathcal{P}^i(k)} {}^{\ell}a^i_{m+1}\}
\end{aligned}
$$

---

*Read: " ℓ is indistinguishable from k."

Since i is informed only of some $\mathcal{P}^i(k) \in \mathcal{P}^i$, he must bid within*
$\min_{\ell \in \mathcal{P}^i(k)} {}^{\ell}a^i_{m+1}$. The reason for the two equalities is clear. Note that
S^i is an $|\mathcal{P}^i|sm$-dimensional set if i is moneyed. Thus the lack of information of a moneyed trader shows up in a reduction of the dimension of his strategy set.

The outcome of the game engendered by a particular bid $b \in S$ is determined in three simple steps. First we calculate a price, ${}^k_{\ell}p_j$, for commodity $j\ell$ ($j \in I_m$, $\ell \in I_s$) in state k , by dividing the amount bid by the amount supplied. Thus:

$$ {}^k_{\ell}p_j = \overline{{}^k_{\ell}b_j} / \overline{{}^{\ell}a_j} \ . $$

Next we calculate the final allocation that results when the bids are executed. Letting ${}^k x^i_j$ stand for the allocation of $j \in I_m$ to trader $i \in I_n$ in state $k \in I_s$, we have

$$ {}^k x^i_j = {}^k\xi^i_j(b) = \begin{cases} {}^k_k b^i_j / {}^k_k p_j & \text{if } {}^k_k p_j > 0 \,, \quad j \in I_m \\ 0 & \text{if } {}^k_k p_j = 0 \,, \quad j \in I_m \\ {}^k a^i_j - \overline{{}^k b^i} + \sum_{t \in I_m} \sum_{\ell \in I_s} {}^k a^i_t \, {}^k_{\ell}p_t \ . & \end{cases} $$

(Clearly this is nonnegative.) Finally we calculate the final utilities of "payoffs" to the traders:

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*There are conventions other than the "min" convention we use here, for example the bids could be bigger, but would have to be cut back if the individual actually obtained less money.

$$\pi^i(b) = u^i(\xi(b)) = \sum_{k \in I_s} \rho_k {}^k u^i({}^k\xi^i(b)) \ .$$

A Nash Equilibrium (or "N.E.") of this game is defined to be a $b^* \in S$ with the property that, for each $i \in I_n$,

$$\pi^i(b^*) = \max_{b^i \in S^i} \pi^i(b^*|b^i) \ ,$$

where $(b^*|b^i)$ is b^* but with b^{*i} replaced by b^i.

Theorem 1. Assume that there are at least two k-moneyed traders who desire commodity j in state k for any $(k,j) \in I_s \times I_m$. Then a N.E. of Γ exists.

We omit the proof of this because all that it involves is the obvious changes of notation in the proof of Theorem 1 in [7]. The major difficulty is that the payoff functions π^i are discontinuous when ${}_k^k p_j = 0$. To overcome this we consider an ϵ-modified game, ${}^\epsilon\Gamma$, $\epsilon > 0$ which is obtained by assuming that in state k an external agency places a bid of $\epsilon > 0$ in the trading posts of commodity kj for $j \in I_m$. This does not alter the strategy sets of the traders, but it does alter their payoff functions and makes them continuous everywhere. Then a straight-forward use of Kakutani's fixed point theorem yields a N.E., ν^ϵ, of ${}^\epsilon\Gamma$. The conditions that we have imposed ensure enough "competition" in the trading posts kj, $j \in I_m$, in state k, to have the prices ${}_k^k p_j^\epsilon$ at ν^ϵ bounded away from zero. Precisely:* there

*This is proved by repeating the proof of Lemma 2 in [7], with some notational changes.

is a $D > 0$, such that ${}_k^k p_j^\epsilon > D$, for all $\epsilon > 0$, for all $j \epsilon I_m$, and for all $k \epsilon I_s$. Thus the limit of ν^ϵ, $\epsilon \to 0$, is a point of continuity of the payoff functions, and hence is a N.E. of the original game Γ.

3.2. Convergence under Replication

Let us consider a "replication sequence" of economies $\Gamma_1, \ldots, \Gamma_\nu \ldots$ There is a fixed number n of types of traders, characterized by their utilities u^i, endowments a^i, and information P^i. The economy Γ_ν has $t = \nu n$ traders, ν of each type. An N.E. in which traders of the same type choose the same strategies is called type-symmetric (denoted T.S.N.E.) and such an N.E. b can be represented as a vector $\hat{b}$ in $S = \mathop{X}\limits_{i=1}^{n} S^i$, where S^i is the strategy set of trader-type is in Γ_ν. Thus, for each ν, a T.S.N.E. ν^b of $\nu\Gamma$ gives a price and an allocation in Γ_1 via $\nu^{\hat{b}}$. An N.E. at which each trader bids less than the money he has will be called an interior N.E. That a T.S.N.E. always exists is proved in Appendix B of [7]. Conditions that guarantee interiority are discussed in [7]. They require, roughly, that there be "enough" money for all traders in all states, and that all traders desire money.

Given a price vector $p \epsilon \Omega^{sm}$, $p = ({}_1^1p_1, \ldots, {}_1^1p_m, \ldots, {}_s^s p_1, \ldots, {}_s^s p_m)$, we construct a $\tilde{p} \epsilon \Omega^{s(m+1)}$ by putting the "price" of money equal to one in all states.

Given a price vector $\tilde{p} \epsilon \Omega^{s(m+1)}$ we define the budget set of trader i to be

$$B^i(\tilde{p}) = \{x \in \Omega^{s(m+1)} : {}^k_kp \cdot {}^kx + {}^kx_{s(m+1)} \leq {}^k_kp \cdot {}^ka^i + {}^ka^i_{m+1},$$

$$\text{and } {}^k_kp_j \cdot {}^kx_j = {}^{k'}_{k'}p_j \cdot {}^{k'}x_j \text{ whenever } k \sim_i k'\} .$$

The reason for putting ${}^k_kp_j \cdot {}^kx_j = {}^{k'}_{k'}p_j \cdot {}^{k'}x_j$ is that trader i, since he cannot distinguish between k and k', must act (i.e. "bid" in our framework) identically in both states, and therefore the value (under $\tilde{p}$) of ${}^kx_j$ and ${}^{k'}x_j$ must be equal. Then we define competitive prices in the usual way.

We can now report the following result.

Theorem 2. Suppose we have a symmetric, interior sequence $\overline{s}^k$ of N.E. which converges to $\overline{s} \in S$. Let $\tilde{p}$ be the price associated with $\overline{s}$. Then $(\tilde{p},1)$ is competitive for Γ_1, and indeed for each Γ_ν.

Again the proof is completely analogous to the proof of Theorem 2 in [7], so we shall talk our way through it. "At the limit" of $\nu \to \infty$, any particular trader i cannot affect the price by his individual bids, and confronts constant prices ${}^k_\ell p_j$ at each trading post. Thus he maximizes over a "truncated" budget set $\hat{\beta}^i(\tilde{p})$ (where the truncation is due to his cash constraints):

$$\hat{\beta}^i(\tilde{p}) = \{\xi^i(b^i) \in \Omega^{s(m+1)} : \overline{{}^kb^i} \leq \min_{\ell \in \mathcal{P}^i(k)} {}^\ell a^i_{m+1}\} .$$

Here $b^i = ({}^1b^i, \ldots, {}^sb^i)$, ${}^kb^i \in \Omega^{s(m+1)}$, and $\xi^i(b^i)$ is given by:

$${}^k\xi^i_k(b^i) = \begin{cases} {}^k_kb^i_j / {}^k_kp_j & \text{for } j \in I_m \\ {}^ka^i_j - \overline{{}^kb^i} + \sum_{t \in I_m} \sum_{\ell \in I_s} {}^ka^i_t \, {}^k_\ell p_t . \end{cases}$$

Note that the prices are constant, and ${}_k^k p_j$ is positive.

Let us rewrite $\hat{\beta}^i(\tilde{p})$ more transparently as:

$$\{x \in \Omega^{s(m+1)} : {}^k x_j^i = {}_k^k b_j^i / {}_k^k p_j \quad \text{for} \quad j \in I_m ,$$

$$ {}^k x_{m+1}^i = {}^k a_{m+1}^i - \overline{{}^k b^i} + \sum_{t \in I_m} \sum_{\ell \in I_s} {}^k a_t^i \, {}_\ell^k p_t ,$$

$$|\mathcal{P}^i(k)| \sum_{j \in I_m} {}_k^k b_j^i \leq \min_{\ell \in \mathcal{P}^i(k)} {}^\ell a_{m+1}^i ,$$

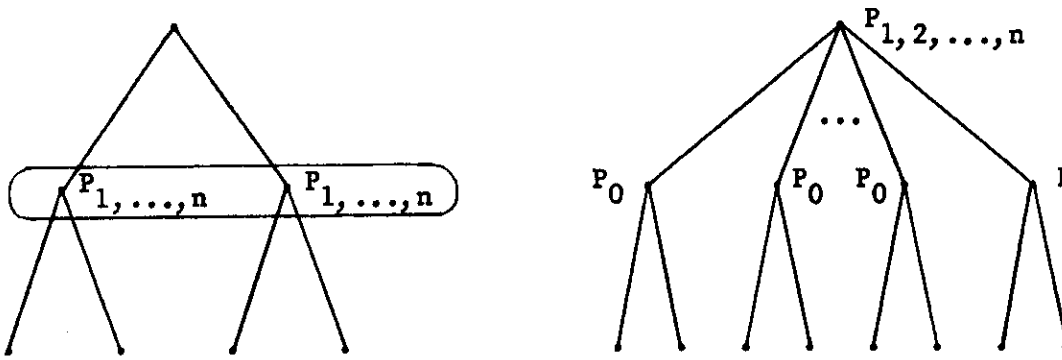
$${}_k^k b_j^i = {}_\ell^\ell b_j^i \quad \text{whenever} \quad k \sim_i \ell \} .$$

The last equality and the inequality reflect the constraints due to his lack of information. But if we are at an <u>interior</u> N.E., then the cash constraint is no longer binding (see proof of Theorem 2 in [7]) and the inequality in the budget set of this is that an interior N.E., the trader's maximum on $\hat{\beta}(\tilde{p})$ is in fact his maximum on $\beta(\tilde{p})$. The theorem now follows by noting that if $\tilde{p}$ is competitive for Γ_1 then it is trivially competitive for each Γ_ν (replicating the allocation of Γ_1).

4. <u>STRATEGIES, FEASIBILITY AND OPTIMALITY</u>

4.1. <u>Anteriority and Preliminarity</u>

A move is said to be anterior to another if it occurs first. It is preliminary to another if the individual currently selecting the move is informed about it. The Arrow-Debreu model to handle contingent commodities can be modelled in extensive form with the random move by nature anterior to the traders moves or vice versa. This is shown in Figure 3a and 3b.



(a) (b)

FIGURE 3

As long as no traders are informed about Nature's move it does not matter whether we model it as occurring before or after the players' moves. If, as in Radner's analysis [9] we wish to consider nonsymmetric information conditions then we must model Nature's move as anterior to those moves of at least some of the traders if we wish to consider only objective probabilities. Radner's interpretation of his model includes subjective probabilities.

4.2. Trade, Communication and Optimality

Suppose that a group of traders can costlessly and timelessly communicate, recontract, bargain and exchange in a completely unrestricted institution-free manner. Furthermore suppose that the group is cooperative. They hence can pool all of their information and all of their goods. They then may distribute their resources in any manner they choose.

It follows immediately that the pooling of information will result in a symmetric distribution of information where states which are known *ex ante* to one can be known to all. For example consider a market with

3 states and 5 traders:

Trader 1 sees (1,2,3)

Trader 2 sees (12) and 3

Trader 3 sees (13) and 2

Trader 4 sees (23) and 1

Trader 5 sees 1, 2 and 3 .

This market treated by Radner would require 7 prices to reflect the wide disparity in information. Viewed cooperatively, as a group the traders know everything.

If we do not specify limits on the method of trade and communication then we may easily interpret the statement that "trader 2 tells trader 1 the outcome of a random variable about which he has been informed" to mean that trader 1 will now have the same information as trader 2. In terms of the extensive form diagrams this communication would convert the information set for trader 1 shown in Figure 1 into two information sets.

When we consider trade as a well defined game the above discussion of communication and information misses a subtle but vital point in the description and definition of a game of strategy. Once the <u>extensive form</u> has been modelled that is the game. A simple example will illustrate the critical distinction which must be made.

Consider two individuals crossing a road, one is blind and the other can see. There are two states of nature: with probability ρ a truck is approaching silently and with $(1-\rho)$ it is not approaching. Unassisted the blind man must employ a strategy independent of the information held by the man who can see. In cooperation the one who can see can guide the blind. This guidance is not the same as restoring the eyesight of the blind man. By the "rules of the game" he is blind. In cooperation he may achieve results as though he were not blind, but just because he is told something by the man who is not blind that does not mean that the blind man can see. For example he could have been told a lie.

4.3. Strategies, Feasibility and Optimality

(a) Unrestricted Trade and Communication

The Pareto optimal surface for a group of traders who can recontract, transfer goods and communicate at any time even given nonsymmetric information is the Pareto optimal surface of the exchange economy with the maximal symmetric refinement of information sets obtained by "pooling information" in the n-person condition; i.e. the Pareto set is defined independent of information.

(b) The Competitive Equilibrium

The Arrow-Debreu methods of dealing with uncertainty do not handle nonsymmetric information but they have more structure than unrestricted trade. Given m commodities and k states a new set of mk commodities are defined. The existence of a price system which clears the markets in trade in these commodities is established. They then show that the resultant distributions of goods and services is Pareto optimal in terms of completely unrestricted trade with equal ignorance for all traders.

(c) The Noncooperative Equilibrium

The models described in Section 1 provided several completely defined market mechanisms for trade. Each model imposed limits on the sets of moves available to each trader. For example in Section 1 with a single compound futures market, a move is a bet on a single black box. With simple futures markets a move becomes two bets on different black boxes.

If all traders can trade in the same markets then they have symmetric limitations to their sets of moves.

A move is different from a strategy. A strategy is an overall plan which is used by an individual to select his moves as a function of his information. Thus with different information the strategy sets of the traders are not symmetric.

Lack of information forces a trader to select the same move in states which he cannot distinguish. Cooperation does not enable him to distinguish those states, however it does enable a partner to tell the trader what to do.

5. OBJECTIVE AND SUBJECTIVE UNCERTAINTY

5.1. Noncooperative Equilibrium

Formally our equilibrium existence proofs still hold if we interpret the probability beliefs of the individuals as subjective probabilities. If we do this we can consider a game which can best be described as a game with misperceived payoffs. This can be reasonably well defined for the single period models dealt with here. When however we try to extend our analysis to multistage markets it is necessary to describe the mechanism whereby individuals change their subjective probabilities.

5.2. Cooperative Solutions and Modelling Problems

An indication of the relatively unsatisfactory way in which information, communication and belief are treated in economic analysis is given when we consider the construction of a cooperative game model of the market described by a noncooperative game with nonsymmetric information. In Sections 4.3 and 4.4 we gave a straightforward mathematical definition of Pareto optimality based upon the specific extensive form description of the markets. This did not permit us to model certain important aspects of human information and communication which are common and can be important. If by the rules of the game A knows something that B does not know, the coalition of A and B acting together may use this extra knowledge, but B cannot be "educated by A ." This point is implicit in the discussion in 4.2 and 4.3.

A more satisfactory model should be able to include the selling of information. [10/]

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